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ITERATIVE SOLUTION OF THE INCREMENTAL PROBLEM FOR ELASTIC–PLASTIC STRUCTURES WITH ASSOCIATED FLOW LAWS

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Abstract—The classical approach to the incremental problem of elastic–plastic structures with elementary work-hardening constituents expressed in terms of an integral equation is converted into a new one free from conditions for the unknown incremental plastic strain.

The paper applies some iterative methods of the integral equation theory and discusses the conditions of uniform convergence of the functions sequence that approximates to the incremental distribution of stresses.

Lastly, it is shown that the iterative methods considered can be extended to the case of structures with work-softening constituents; a condition that usually accompanies the study of such systems coincides with a convergence condition of one of the iterative methods discussed.

1. INTRODUCTION

WORKING from the incremental theory of elastoplasticity, several authors have recently attempted to devise general methods for the study of elastic–plastic structures that would be simpler to apply than the classic variational principles of plasticity theory. Among these are the contributions [1] [2], who formulated a new general principle for perfectly plastic or work-hardening materials.

Parallel studies on the general stability of equilibrium supply sufficient uniqueness and stability conditions of the incremental solution for trusses [3] and for a continuum [4] with unstable constituents (elementary or finite); further [5] they show how the variational principles can be extended to systems with work-softening constituents.

Within this framework the writer recently tackled the problem of stability [6] in terms of an integral equation that had proved rewarding for this purpose. Reference was made, albeit indirectly, to the incremental elastic–plastic problem, though no attempt was made to solve it because of the analytical difficulties involved.

This paper takes up the problem where it was left in [6] and aims to show how, if some restrictive hypotheses are imposed, these difficulties can be overcome by using some well-known properties of integral equation theory in order to generate an iterative procedure for calculating elastic–plastic systems.

The incremental flow laws assumed for the generalized stress and strain components $Q_i(x)$ and $q_i(x)$ of a given three-dimensional continuum* are of the associated type and concavity in yield surface $\Phi(Q_i) = 0$ is not excluded. The incremental plastic strain $\dot{q}_i^p(x)$ is defined as the difference between the actual $\dot{q}_i(x)$ and the $\dot{q}_i^e(x)$ that would apply in the case of elastic behavior.

* Hereafter x will denote the coordinates x, y, z of a given point in the three-dimensional body. As usual, it is assumed that the properties of the materials are time-independent and t stands for a given increasing function of time that represents the succession of the states of the system. The symbol $\cdot$ denotes the differential with respect to t.

The constitutive incremental laws are summed up in the usual form:

$$\dot{q}_i^p(x) = \frac{\alpha(x)}{\mu(x)} \frac{\partial \Phi}{\partial Q_i}(x) \, . \, \frac{\partial \Phi}{\partial Q_j}(x) \, . \, \dot{Q}_j(x) = \dot{\lambda}(x) \, . \, \frac{\partial \Phi}{\partial Q_i}(x) \tag{1}$$

where in the case of work-hardening materials ($\mu(x) > 0$)

$$\begin{aligned} &\alpha(x) = 0 \quad \text{for } \Phi < 0 \quad \text{or for } \Phi = 0 \quad \text{and} \quad \dot{\Phi} = \dot{Q}_i \frac{\partial \Phi}{\partial Q_i} \leq 0 \\ &\alpha(x) = 1 \quad \text{for } \Phi = 0 \quad \text{and } \dot{\Phi} > 0. \end{aligned} \tag{2}$$

In other words, for the time being perfectly plastic or work-softening materials such as were considered in the broader class of behaviors dealt with in [6] are excluded. This does not mean that the reasoning outlined here cannot be applied to the latter types of behavior in certain conditions, which will be detailed later.

On the broader assumptions considered in [6], and hence also on the more restricted ones indicated here, the problem of incremental response due to an increment $\dot{F}_i(x)$ of the loads starting from a known configuration with extent V_p of the plastic region is reduced to the solution of the integral equation in $\dot{\lambda}(x)$:

$$\dot{\lambda}(x) - \int_{V_p} \frac{\alpha(x)}{\mu(x)} \, . \, Z(x, x') \, . \, \dot{\lambda}(x') \, \mathrm{d}V' = \frac{\alpha(x)}{\mu(x)} f(x) \tag{3}$$

on the additional condition, given the essentially positive character of $\dot{\lambda}(x)$:

$$\dot{\lambda}(x) \geq 0 \tag{4}$$

as well as in the presence of the alternative

$$\alpha(x) = \begin{cases} 0 \\ 1 \end{cases} \tag{5}$$

for every x in V_p according to whether or not elastic unloading ($\alpha(x) = 0$) occurs.

In (3) the function $f(x)$ represents, to within the factor $\partial\Phi/\partial Q_i$, the increment $\dot{Q}_i^e(x)$ of the stress distribution due to the increment of the loads $\dot{F}_i(x)$ assuming that the materials behave elastically; the second term of the first member represents the contribution to the stress distribution made by the effective plastic strains $\dot{q}_i^p(x)$ seen as dislocations to which the body is subject if the material behave elastically, since $Z(x, x')$, to within the factor $(\partial\Phi/\partial Q_i)(x) \, . \, (\partial\Phi/\partial Q_j)(x')$, is the influence function $z_{ij}(x, x')$ of a unit dislocation $\dot{q}_j^p(x')$ in x' on the ith elastic stress component at x.

Starting with this approach, it is shown that the problem can be transformed into a new one (in the classic form, i.e. before conditions for the unknown function $\dot{\lambda}(x)$) for which the known iterative criteria of solution apply.

In particular for the continuous one-dimensional case without interactions, by means of iteration we find for a particular case the solution of the incremental problem.

The numeric solution thus obtained is then compared with the solution that with reference to the first step of the loading program can be obtained by means of the analogic method proposed in [7].

Lastly, the application of the method to the case of work-softening is discussed and the possibility of extending it to the case of perfectly plastic materials is suggested.

2. EVOLUTION OF THE PLASTIC REGION

Assumptions (1) and (2) regarding the constitutive laws presuppose load increments that are small enough not to bring points that are initially in the elastic range into the plastic range. As a consequence of the above assumptions the integration domain of (3) coincides with the initial plastic region V_p. In other words, during a step [1] in the loading program it is assumed that the plastic region either remains constant (which would happen for $\alpha(x) = 1$ for every x in V_p) or decreases (by $\dot{V}_{p-e}$ if $\dot{V}_{p-e}$ represents the region of V_p in which $\alpha(x) = 0$), but anyway can never increase.

Similar behavior may occur in the discrete case [2] (that is for trusses or continua divided into finite parts, etc.) where the variations in region V_p are discontinuous, V_p remaining constant until, after a succession of steps in the loading program, any (finite) element reaches the yield limit.

This type of behavior does not usually occur in a continuum since the variation in V_p is generally* a continuous function of the loads. The previous assumption regarding the extent of V_p may still be made, however, as the variation $\dot{V}_{e-p} - \dot{V}_{p-e}$ of V_p ($\dot{V}_{e-p}$ is the region in which $\alpha(x) = 1$) cannot be other than infinitesimal (at least of the 1st order with $\dot{\lambda}(x)$) and so in (3) can supply contributions of an order above the 1st that can be ignored in a first-order approach.

Notwithstanding this, a few difficulties may still arise. In the particular case of an initial situation with a finite number of points at the yield limit (and hence $V_p = 0$), (3) supplies for $\dot{\lambda}(x)$ the expression

$$\dot{\lambda}(x) = \frac{\alpha(x)}{\mu(x)} \cdot \dot{f}(x) \tag{6}$$

obviously wrong for a general load increment and anyway unsuitable for determining the incipient formation of plastic region $\dot{V}_{e-p}$; it is thus impossible to describe the change of the plastic strains in the successive steps since the extent of the integration domain V_p of (3) always remains zero.

All this is the result of ignoring in (3) the term

$$\int_{\dot{V}_{e-p}} \frac{\alpha(x)}{\mu(x)} \cdot Z(x, x')\dot{\lambda}(x')\, \mathrm{d}V'$$

because it is of the second order, although it alone is responsible for the increment of V_p.

The reference to the last case suggests that the change of V_p cannot be fully identified unless we take an approach including in (3) infinitesimal contributions of a higher order than the 1st; by this means the increments of V_p will occur within each individual step and not only in the succession of steps in the loading program, as is the case with the 1st order approach.

All this constitutes a closer approximation to the real behavior of structures and makes necessary a more precise definition of the constitutive incremental law of the material which, as stated in the terms indicated in (1) and (2), does not cover the case of increments in V_p within a step starting from situations in the elastic range.

* The case of finite variations in V_p for an assigned infinitesimal load increment is explicitly excluded.

Writing equation (1) in the form

$$\dot{q}_i^p(x) = \frac{\partial \Phi}{\partial Q_i}(x) \, . \, g[\dot{\Phi}(x)] \tag{7}$$

where $Q_{i0}(x)$ denotes the initial stress distribution, we define function $g[\dot{\Phi}(x)]$ as follows:

$$\text{for } \Phi[Q_{i0}(x)] \geq 0 \begin{cases} \text{if } \dot{\Phi}(x) \leq 0 & g[\dot{\Phi}(x)] = 0 \quad (8) \\ \text{if } \dot{\Phi}(x) > 0 & g[\dot{\Phi}(x)] = \dfrac{\dot{\Phi}(x)}{\mu(x)} \quad (9) \end{cases}$$

$$\text{for } \Phi[Q_{i0}(x)] < 0 \begin{cases} \text{if } \dot{\Phi}(x) \leq 0 \\ \text{or } \dot{\Phi} > 0 \text{ and } \Phi[Q_{i0}(x)+\dot{Q}_i(x)] < 0 & g[\dot{\Phi}(x)] = 0 \quad (10) \\ \text{if } \dot{\Phi}(x) > 0 & g[\dot{\Phi}(x)] = \dfrac{\dot{\Phi}(x)}{\mu(x)} \\ \text{and } \Phi[Q_{i0}(x)+\dot{Q}_i(x)] > 0 & \quad + \bar{f}[Q_{10}(x), \ldots Q_{i0}(x), \ldots] \quad (11) \end{cases}$$

where $\bar{f}[Q_{10}(x), \ldots Q_{i0}(x), \ldots]$ is a known function (depending upon the type of structure and material) of point A in the stress space representing the initial stress distribution at x and depending on the evolution, within the step, of stresses $Q_{i0}(x)$ at the final value $Q_{i0}(x)+\dot{Q}_i(x)$ (Fig. 1, where $\vec{\dot{Q}}$ and $\vec{\dot{q}}$ are the incremental stress and strain vectors of components $\dot{Q}_1, \ldots \dot{Q}_i$, $\dot{q}_1 \ldots \dot{q}_i$).

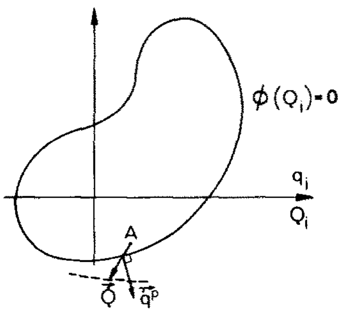

FIG. 1.

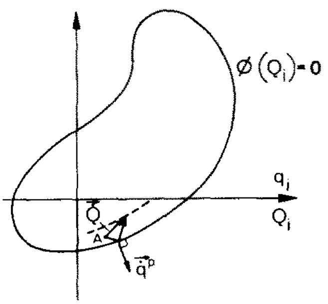

FIG. 2.

(11) thus allows for the transition of points in the elastic range to the plastic range during the step. Increments $\dot{F}_i(x)$ are, however, assumed to be such that during the interval of time dt relating to a given step in the loading program the forces $F_i(x)$ and the generalized stress and strain components Q_i and q_i vary proportionally; it is always possible to ensure this by acting on the amplitude of the load increments assigned.

On the basis of these new assumptions, with equation (3) written in the form

$$\dot{Q}_i(x) - \int_{V_p - \dot{V}_{p-e} + \dot{V}_{e-p}} z_{ij}(x, x') \, . \, \dot{q}_j^p(x') \, \mathrm{d}V' = \dot{Q}_i^f(x) \tag{12}$$

where $\dot{V}_{p-e}$ and $\dot{V}_{e-p}$ are the regions in transition from the elastic to the plastic range and vice versa, by means of (8), (9), (10) and (11) it is possible to pass from the incremental problem set in the terms of (3), (4) and (5) to the problem expressed as a single nonlinear equation:

$$\dot{\Phi}(x) - \int_V Z(x, x') \,.\, g[\dot{\Phi}(x')] \, \mathrm{d}V' = f(x) \tag{13}$$

in the unknown function $\dot{\Phi}(x)$ without limitation of sign.

As to the domain of integration, it may be formally extended to the whole volume V of the continuum instead of to the region $V_p - \dot{V}_{p-e} + \dot{V}_{e-p}$ since the integral will supply a non zero contribution only in the cases (9) and (11), which form part of the unknown region $V_p - \dot{V}_{p-e} + \dot{V}_{e-p}$.

3. SOLUTION OF THE INCREMENTAL ELASTIC PLASTIC PROBLEM

The transformation of the problem set in terms of (3), (4) and (5) into the form (13) (that is free from conditions for the unknown function $\dot{\Phi}(x)$) proves to be especially useful in solving the problem since with $h(x')$ representing a function such that*

$$f(x) = \int_V Z(x, x') \,.\, h(x') \, \mathrm{d}V' \tag{14}$$

and with

$$m[x', \dot{\Phi}(x')] = g[\dot{\Phi}(x')] + h(x') \tag{15}$$

equation (13) becomes

$$\dot{\Phi}(x) - \int_V Z(x, x') \,.\, m[x', \dot{\Phi}(x')] \, \mathrm{d}V' = 0 \tag{16}$$

which falls within the type of nonlinear integral equation in the Hammerstein's standard form. The theory of integral equations supplies some weak sufficient conditions of uniqueness of solution involving the kernel $-Z(x, x')/\sqrt{[\mu(x)]} \,.\, \sqrt{[\mu(x')]}$ and the function $m(x, u)$. Precisely the kernel must satisfy the inequality:

$$\|Z\|^2 = \int\int_V Z^2(x, x') \, \mathrm{d}V \, \mathrm{d}V' \leq N^2 \tag{17}$$

* It is always possible to find a function $h(x')$ such that (14) holds good whatever $f(x)$ may be. Indeed, because of the presence of influence function $Z(x, x')$, $h(x')$ acquires the meaning of distribution of dislocations imposed on the elastic body equal to the elastic strains due to the loads $\dot{F}_i(x)$. In other words, referring back to the splitting of the stress $\dot{Q}_i(x)$ into the sum of elastic stresses $\dot{Q}_i^f(x)$ and the self-stresses $Q_i^s(x)$, we may write it as follows:

$$f(x) = \frac{\partial \Phi}{\partial Q_i}(x) \,.\, \dot{Q}_i^f(x) = \frac{\partial \Phi}{\partial Q_i}(x) \,.\, \int_V z_{ij}(x, x') \,.\, \dot{q}_j^f(x') \, \mathrm{d}V'$$

and hence

$$h(x') = \dot{q}_j^f(x') \,.\, \left[\frac{\partial \Phi}{\partial Q_j}(x')\right]^{-1}$$

with N constant, which is certainly verified;* the function $m(x, u)$ must satisfy the conditions of the first Hammerstein's theorem of uniqueness for equations with semidefinite positive kernel. In the present case these conditions are fulfilled if the work-hardening coefficient $\mu(x)$ is positive.

This latter requirement is covered by the hypothesis of first section; thus the uniqueness of solution is ensured.

With regard to the solution of (16), several of the numerous iterative methods frequently employed in the study of nonlinear integral equations may be used. They are related formally to the known methods of approximate solution of algebraic and transcendental equations, i.e. to the simple iteration or the Raphson–Newton method.

By using these methods and writing (16) in the form

$$\dot{\Phi}(x) = F[\dot{\Phi}(x)] = \dot{\Phi}(x) - G[\dot{\Phi}(x)] \tag{16'}$$

the two fundamental approximate methods mentioned above lead respectively to the following iterative relations

$$\dot{\Phi}_{n+1}(x) = F[\dot{\Phi}_n(x)] \tag{18}$$

$$\dot{\Phi}_{n+1}(x) = \dot{\Phi}_n(x) - \frac{\dot{\Phi}_n - F[\dot{\Phi}_n(x)]}{1 - F'[\dot{\Phi}_n(x)]} = \dot{\Phi}_n(x) - \frac{G[\dot{\Phi}_n(x)]}{G'[\dot{\Phi}_n(x)]} \tag{19}$$

with

$$F'[\dot{\Phi}_n(x)] = \left[\frac{\partial F[\dot{\Phi}(x)]}{\partial \dot{\Phi}(x)}\right]_{\dot{\Phi} = \dot{\Phi}_n}$$

and thence to the sequence

$$\dot{\Phi}_1(x), \dot{\Phi}_2(x), \ldots, \dot{\Phi}_n(x), \ldots. \tag{20}$$

Many sufficient conditions for the uniform convergence of the sequence (20) are known; however, these are very restrictive conditions, which kernel $Z(x, x')/\sqrt{[\mu(x)]}.\sqrt{[\mu(x')]}$ or its eigenvalues Γ_i must satisfy. With reference to (18), from the fundamental theorem of functional analysis of iterative processes [9] [12] we can easily deduce the conditions

$$\left|\frac{Z(x, x')}{\sqrt{[\mu(x)]} \cdot \sqrt{[\mu(x')]}}\right| < 1 \tag{21}$$

$$|\Gamma_i| > 1 \tag{22}$$

that are singly sufficient for convergence. They, however, express over-restrictive conditions that do not generally occur since they set limitations to the values of the work-hardening coefficient $\mu(x)$, which is assumed to be positive arbitrary. This explains why we cannot for the moment make systematic use of the iteration method (18), which has the advantage of being the simpler of the two methods considered, even though convergence is not always rapid.

* Indeed, it may always be said that: $\iint_V Z(x, x')\, \mathrm{d}V\, \mathrm{d}V' \leq M(.)$ with M constant when, to within the sign, the first member is interpreted as strain energy due to unit dislocations imposed on the body supposed to have elastic behavior. Lastly, (.) leads to (17).

By the Raphson–Newton method, on the other hand, (19) may be written in the following way

$$\dot{\Phi}(x)-\dot{\Phi}_{n+1}(x) \simeq \tfrac{1}{2}[\dot{\Phi}(x)-\dot{\Phi}_n(x)]^2 \cdot \frac{G''[\dot{\Phi}(x)]}{G'[\dot{\Phi}(x)]} \tag{23}$$

from the recurrent application of which it follows that

$$\dot{\Phi}(x)-\dot{\Phi}_{n+1}(x) \simeq \{a_1[\dot{\Phi}(x)-\dot{\Phi}_1(x)]\}^{2^{(n+1)}}/a_1 \tag{24}$$

which guarantees uniform convergence of sequence (20) on the exact solution for any value of a_1 (that is for any kernel $Z(x,x')/\sqrt{[\mu(x)]}\cdot\sqrt{[\mu(x')]}$ whatsoever), provided that function $\dot{\Phi}_1(x)$ of first approximation be sufficiently close to the exact solution $\dot{\Phi}(x)$ so that $a_1[\dot{\Phi}(x)-\dot{\Phi}_1(x)]$ is less than unity.

This only condition is generally rather restrictive since a function $\dot{\Phi}_1(x)$ that approximates to $\dot{\Phi}(x)$ is most unlikely to be known. In this case the assumptions concerning the size of the increments $\dot{F}_i(x)$ and the evolution of elastic–plastic region allow us to conceive the known incremental elastic solution to the problem complies with the required conditions of approximation of the solution $\dot{\Phi}(x)$. Should this not be so for assigned values of $\dot{F}_i(x)$, there would still be no difficulty in satisfying the above condition, as it would always be possible, by checking the distribution of the additional loads, to ensure that the incremental elastic solution was exceedingly close to the effective elastic–plastic solution.

And so if we once again express (16) in the form (13) and apply the two iterative methods considered, (18) and (19) supply the recurrent relations:

$$\dot{\Phi}_{n+1}(x) = \int_V Z(x,x')g[\dot{\Phi}_n(x')]\,\mathrm{d}V'+f(x) \tag{25}$$

$$\dot{\Phi}_{n+1}(x) = -(B-1)\dot{\Phi}_n(x)+B\left\{\int_V Z(x,x')\cdot g[\dot{\Phi}_n(x')]\,\mathrm{d}V'+f(x)\right\} \tag{25$'$}$$

where $B = 1/(1-A)$ and

$$A = \left[\frac{\partial}{\partial\dot{\Phi}}\int_{V_p-\dot{V}_{p-e}+\dot{V}_{e-p}} Z(x,x')g[\dot{\Phi}]\,\mathrm{d}V'\right]_{\dot{\Phi}=\dot{\Phi}_n}$$

B cannot be evaluated a priori because the extent of $V_p-\dot{V}_{p-e}+\dot{V}_{e-p}$ depends on the unknown $\dot{\Phi}(x)$; it can therefore be calculated by approximation from the expressions of B_n obtained by substituting for $V_p-\dot{V}_{p-e}+\dot{V}_{e-p}$ the region V_p^n that is defined by each of the functions $\dot{\Phi}_n(x)$ approximating to the exact solution.

In the particular case of $V_p = 0$ in a second order approach (or in the case of $V_p \neq 0$ for a first order approach) we would point out that the terms containing A may be rigorously disregarded and so (25′) may be converted into the simpler recurrent relation (25).

Thus the earlier remarks regarding the poor changes of using (25) no longer stand; and the same applies to conditions (21) and (22), now rendered superfluous by the proven convergence of the method of Raphson–Newton, from which the iteration method can now be derived.

Lastly, we would point out that for $\dot{\Phi}_1(x) \equiv 0$ the convergence of the sequence (20) is still assured by the fact that from (25′) we obtain:

$$\dot{\Phi}_2(x) = f(x) = \frac{\partial\phi}{\partial Q_i}(x)\cdot\dot{Q}_i^f(x)$$

which coincides with the incremental elastic solution. Hence the sequence (20) is bound to converge on the exact solution $\dot{\Phi}(x)$ of the problem.

4. ONE-DIMENSIONAL CASE

It was shown in [6] that a typical feature of this particular case is the degeneracy of kernel $Z(x, x')$, i.e. the separability of the dependence of $Z(x, x')$ on the variables x and x' in the form of the sum of products of functions of x and x' only according to the relation:

$$Z(x, x') = \sum_{k=1}^{r} H_k(x) \,.\, G_k(x') \tag{26}$$

where r is finite and equal to the number of redundant reactions of the system. Referring back to (25') we may write:

$$\dot{\Phi}_{n+1}(x) = -(B-1)\dot{\Phi}_n(x) + B\left(\sum_{k=1}^{r} H_k(x) \int_V G_k(x')g[\dot{\Phi}_n(x')]\,\mathrm{d}V' + f(x)\right) \tag{27}$$

and so the iterative method may be summed up in the recurrent equation:

$$\dot{\Phi}_{n+1}(x) = -(B-1)\dot{\Phi}_n(x) + B\left(\sum_{k=1}^{r} H_k(x) \,.\, A_{kn} + f(x)\right) \tag{28}$$

assuming that:

$$A_{kn} = \int_V G_k(x')g[\dot{\Phi}_n(x')]\,\mathrm{d}V'. \tag{29}$$

The method is thus reduced to the calculation of constants A_{kn} only.

Because of the physical meaning attached in [6] to $H_k(x)$ i.e. the influence function of the kth redundant reaction on the ith generalized stress component at x, coefficients A_{kn} take on the meaning of redundant reactions (of nth approximation). It is thus shown that the iterative method arrives at the solution $\dot{\Phi}(x)$ by successive approximations of the self-stress distribution to be added as a "corrective" to the stress distribution $f(x)$ due to $\dot{F}_i(x)$ in a body supposed to have an elastic behavior.

All that remains now is to state the incremental law governing the generalized stress and strain components $g[\dot{\Phi}(x)]$. Assuming beams or systems of straight beams subjected to transverse loads, as usual in the case of bent beams in the elastic range, strains due to the shear forces are ignored and it is assumed that there is no interaction between the shear forces and bending moments.

The generalized stress and strain components are thus reduced to the bending moment $Q(x)$ and the curvature $q(x)$.

For the elementary constituent (beam element $\mathrm{d}x$) it is assumed that the elastic–plastic law with linear work-hardening holds and $\bar{Q}$, EJ, $\bar{\mu}(x)$ denote respectively the bending moment at the elastic limit, the bending stiffness in the elastic range and the bending stiffness in the plastic range.*

Thus, on the assumption that the material behaves symmetrically under tensile and compressive stresses, expression (1) becomes:

$$|Q(x)| - \bar{Q}(x) = 0 \tag{30}$$

* The relation between these coefficients and the work-hardening coefficient $\mu(x)$ is:

$$\frac{1}{\mu(x)} = \left(\frac{1}{\bar{\mu}(x)} - \frac{1}{EJ}\right)$$

and the constitutive law (8), (9), (10), (11) with $Q_0(x)$ denoting the stress distribution of any given section x in the initial situation* will become:

for $Q_0(x) \geq 0$:

if $Q_0(x) \geq \bar{Q}(x)$

$$\dot{q}^p(x) = +\frac{1}{2}\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)\{\dot{Q}(x)+|\dot{Q}(x)|\} = \begin{cases} 0 & \text{for } \dot{Q}(x) \leq 0 \quad \text{(31a)} \\ +\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)\dot{Q}(x) & \text{for } \dot{Q}(x) > 0 \quad \text{(31b)} \end{cases}$$

if $Q_0(x) < \bar{Q}(x)$

$$\dot{q}^p(x) = +\frac{1}{2}\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right) \times \{\dot{Q}(x)+Q_0(x)-\bar{Q}(x)+|\dot{Q}(x)+Q_0(x)-\bar{Q}(x)|\} =$$

$$= \begin{cases} 0 \text{ for } \dot{Q}(x) \leq 0 & \text{and for } \dot{Q}(x) > 0 \text{ but } \dot{Q}(x)+Q_0(x)-\bar{Q}(x) \leq 0 \quad \text{(32a)} \\ +\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)[\dot{Q}(x)+Q_0(x)-\bar{Q}(x)] & \text{for } \dot{Q}(x) > 0 \text{ but } \dot{Q}(x)+Q_0(x)-\bar{Q}(x) > 0 \quad \text{(32b)} \end{cases}$$

for $Q_0(x) < 0$:

if $Q_0(x) < -\bar{Q}(x)$

$$\dot{q}^p(x) = +\frac{1}{2}\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)\{\dot{Q}(x)-|\dot{Q}(x)|\} = \begin{cases} 0 & \text{for } \dot{Q}(x) \geq 0 \quad \text{(33a)} \\ \left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)\dot{Q}(x) & \text{for } \dot{Q}(x) < 0 \quad \text{(33b)} \end{cases}$$

if $Q_0(x) > -\bar{Q}(x)$

$$\dot{q}^p(x) = +\frac{1}{2}\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right) \times\{\dot{Q}(x)+Q_0(x)+\bar{Q}(x)-|\dot{Q}(x)+Q_0(x)+\bar{Q}(x)|\} =$$

$$= \begin{cases} 0 \quad \text{for } \dot{Q}(x) \geq 0 & \text{and for } \dot{Q}(x) < 0 \text{ but } \dot{Q}(x)+Q_0(x)+\bar{Q}(x) \geq 0 \quad \text{(34a)} \\ +\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)[\dot{Q}(x)+Q_0(x)+\bar{Q}(x)] & \text{for } \dot{Q}(x) < 0 \text{ and } \dot{Q}(x)+Q_0(x)+\bar{Q}(x) < 0. \quad \text{(34b)} \end{cases}$$

* The sagging bending moment $Q(x)$ is assumed to be positive. So:

$$\frac{\partial \Phi}{\partial Q}(x) = \pm 1$$

according to whether $Q(x)$ is positive or negative respectively. With regard to the work-hardening coefficient $\mu(x)$ we shall assume hereafter that it is constant as it frequently occurs in straight beams. No difficulty would arise if we were to assume both μ and EJ as x dependent.

It follows from the flow law stated above that the incremental plastic strain is related to the stress increment $\dot{Q}(x)$ by only one equation:

$$\begin{aligned}\dot{q}^p(x) = {} & +\frac{1}{2}\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)\left\{\dot{Q}(x)+\frac{1}{2}\left[Q_0(x)-\frac{Q_0(x)}{|Q_0(x)|}\bar{Q}(x)-\frac{Q_0(x)}{|Q_0(x)|}\cdot\left|Q_0(x)-\frac{Q_0(x)}{|Q_0(x)|}\cdot\bar{Q}(x)\right|\right]\right.\\ & \left.+\frac{Q_0(x)}{|Q_0(x)|}\cdot\left|\dot{Q}(x)+\frac{1}{2}\left[Q_0(x)-\frac{Q_0(x)}{|Q_0(x)|}\cdot\bar{Q}(x)-\frac{Q_0(x)}{|Q_0(x)|}\left|Q_0(x)-\frac{Q_0(x)}{|Q_0(x)|}\cdot\bar{Q}(x)\right|\right]\right|\right\} \qquad (35)\\ = {} & \dot{q}^p[x,\dot{Q}(x)]\end{aligned}$$

and so by (25'), since $[(\partial\Phi/\partial Q)(x)]^2 = 1$ yields the recurrent relation

$$\dot{Q}_{n+1}(x) = -(B-1)\dot{Q}_n(x)+B\left(\int_V z(x,x')\dot{q}^p[x',\dot{Q}_n(x')]\,\mathrm{d}x'+\dot{Q}^f(x)\right). \qquad (36)$$

In our assumptions, (35) sums up the incremental flow law, whilst (31), (32), (33) and (34) express it in various particular cases. They specify the usual elastic-work-hardening behavior and moreover the dependence of the plastic strains upon the initial stress distribution $Q_0(x)$ according to the new hypotheses on the evolution of plastic region V_p.

5. EXAMPLE OF APPLICATION

We shall now take one of the cases discussed in section 2 and determine the incremental solution when the plastic region V_p in the initial situation is zero and when to follow the evolution of V_p, the increment within each step must be evaluated.

Let us consider for this purpose a straight beam of constant cross-section and length l clamped at one end and supported at the other. Acting on it is a uniformly distributed load of constant intensity:

$$\bar{p} = 8\bar{Q}/l^2 \qquad (37)$$

Starting from this situation (which we shall call hereafter the initial situation), characterized by a stress distribution in the elastic range for all sections of beam except the clamped section, we aim to determine the response of the system in terms of the corresponding increment $\dot{Q}(x)$ and $\dot{q}(x)$ of the stress and strain distribution due to an infinitesimal increment $\dot{p}(x)$ = cost of the loads.

The example has been chosen in order that it may be studied also by a known analogic method [7] so that we may obtain useful reference values for the results obtained by the iterative method. It should be noted that the analogic method is of limited applicability in this type of problem in that it assumes a holonomic stress–strain law, which excludes elastic unloading.

A limitation of this kind is obviously inadmissible when for $V_p > 0$ the elastic unloading is not small and precludes its general application. In the present case with an initial limit situation and confining our attention to the first step in the loading program, we can exclude the presence of unloading.* The assumptions underlying the iterative

* This is because in the time interval dt relating to a given step in the loading program it is assumed that forces F_i and components Q_i, q_i will vary proportionally. In the first step this assumption excludes the elastic unloading because, as $V_p = 0$, this would occur as a result of previous plasticization during the step and with a change in the ratios of Q_i to q_i. In the second and subsequent loading steps, however, unloading may occur right from the beginning of the step starting from existing plastic stress distributions ($V_p \neq 0$) without affecting the above ratio in the course of the step. In this case the unloading that has occurred must be taken into consideration and the analogic method cannot be applied.

method thus specialize and fall within the more restrictive limitation of the analogic method, which now can be used directly; thus for the first loading step only, the two approaches (iterative and analogic) lead to the same results.

By the analogic method, having assumed the origin of the axis x at the simply supported end of the beam, and $p(x) = \bar{p}+\dot{p} = \text{cost} = p$ and denoting as

$$Q(x) = Q_0(x)+\dot{Q}(x) = \frac{plx}{2}-\frac{px^2}{2}-\frac{Q_i x}{l} \tag{38}$$

the equilibrated distribution of the bending moments where $Q_i = \bar{Q}+\dot{Q}(l)$ is the unknown redundant reaction at the clamped end, solution $\dot{Q}(x)$ is obtained, as is known [7], from the equation:

$$-\frac{1}{EJ}\int_0^{\varepsilon l}\left(\frac{plx}{2}-\frac{px^2}{2}-\frac{Q_i x}{l}\right)x\,dx-\frac{1}{\bar{\mu}}\int_{\varepsilon l}^{l}\left(\frac{plx}{2}-\frac{px^2}{2}-\frac{Q_i x}{l}+\bar{Q}\right)x\,dx+\frac{\bar{Q}}{EJ}\int_{\varepsilon l}^{l}x\,dx = 0 \tag{39}$$

which expresses the compatibility condition as a condition of equilibrium of the auxiliary beam, εl being the extent of the elastic region up to the cross-section in which

$$Q(\varepsilon l) = -\bar{Q}. \tag{40}$$

In virtue of (40) we obtain from (39) a fifth degree algebraic equation in ε whose solution physically feasible $\bar{\varepsilon}$ supplies via (38) the incremental moment at the clamped end

$$\dot{Q}(l) = (1-\bar{\varepsilon})\left(\frac{pl^2}{2}+\frac{\bar{Q}}{\bar{\varepsilon}}\right) \tag{41}$$

and so lastly the unknown incremental distribution, assuming $\gamma = \dot{p}/\bar{p}$

$$\dot{Q}(x) = \left[4-\frac{\dot{Q}(l)}{\gamma\bar{Q}}\right].\frac{\dot{p}lx}{8}-\frac{\dot{p}x^2}{2} \tag{42}$$

In particular, assuming:

$$\frac{EJ}{\bar{\mu}} = 4 \tag{43}$$

$$\gamma = 2\,.\,10^{-2} \tag{44}$$

it follows that

$$\dot{Q}(l) = 0{\cdot}019658714441\bar{Q}. \tag{45}$$

At the stage of numeric application of the iterative procedure we note first of all that as

$$Q_0(x) = \tfrac{3}{8}\bar{p}lx-\bar{p}\frac{x^2}{2} \tag{46}$$

$$\dot{Q}^f(x) = \tfrac{3}{8}\dot{p}lx-\dot{p}\frac{x^2}{2} \tag{47}$$

equation (13) may be written in the form:

$$\dot{Q}(x) = \int_0^{\frac{3}{4}l} z(x,x')\frac{1}{2}\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)\{\dot{Q}(x')+Q_0(x')-\bar{Q}+|\dot{Q}(x')+Q_0(x')-\bar{Q}|\}\,\mathrm{d}x' \\ +\int_{\frac{3}{4}l}^{l} z(x,x')\frac{1}{2}\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)\{\dot{Q}(x')+Q_0(x')+\bar{Q}-|\dot{Q}(x')+Q_0(x')+\bar{Q}|\}\,\mathrm{d}x'+\dot{Q}^f(x). \tag{48}$$

Whatever the unknown distribution $\dot{Q}(x)$ may be, given the stress distribution $Q_0(x)$ of the cross-sections for $0 < x < \frac{3}{4}l$ at a finite "distance" from the yield limit (the minimum occurs at $x = 3l/8$), the first integral in the second member of (48) is bound to yield a zero contribution, since it falls within the case (32a); the same applies to the second integral in relation to the case (34a), except that for the region $\dot{V}_{e-p}$ at the end of the step where, because of transitions of cross-sections from the elastic range to the plastic range, in virtue of (34b) there must be a non zero contribution. Equation (48) thus becomes:

$$\dot{Q}(x) = \int_{\dot{V}_{e-p}} z(x,x')\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)[\dot{Q}(x')+Q_0(x')+\bar{Q}]\,\mathrm{d}x'+\dot{Q}^f(x) \tag{49}$$

where $\dot{V}_{e-p}$ is the unknown to be determined along with function $\dot{Q}(x)$.

Referring back to (25), which is a particular form of (25′) in the case under consideration, from (36) we obtain:

$$\dot{Q}_{n+1}(x) = \int_{V_p^n} z(x,x')\left(\frac{1}{\bar{\mu}}-\frac{1}{EJ}\right)[\dot{Q}_n(x')+Q_0(x')+\bar{Q}]\,\mathrm{d}x'+\dot{Q}^f(x) \tag{50}$$

where starting from the situation $\dot{Q}_1(x) \equiv 0$ we let V_p^n stand for the region in which $\dot{Q}_n(x')$ comply with (34b).

Having substituted in (50) the expression of the influence function $z(x, x')$ of a positive unit dislocation $\dot{q}(x')\,.\,\mathrm{d}x'$ at x' on the bending moment at x, i.e.

$$z(x,x') = -\frac{3EJ}{l^3}xx', \tag{51}$$

it may be noted that at each stage of iteration the integral in the second member of (50) supplies a contribution of the type $a_n\dot{p}lx/8$ correcting $\dot{Q}^f(x)$. Assuming

$$a_n = 1-\frac{b_n}{\gamma\bar{Q}} \tag{52}$$

the expression of the incremental response $\dot{Q}_{n+1}(x)$ for substitution in the right hand side of (50) for the continuation of the cycle takes the form

$$\dot{Q}_{n+1}(x) = \left(4-\frac{b_n}{\gamma\bar{Q}}\right)\frac{\dot{p}lx}{8}-\frac{\dot{p}x^2}{2} \tag{53}$$

where, in virtue of (42) coefficients b_n take on the meaning of nth approximation values of the incremental moment at the clamped end $\dot{Q}(l)$. Given the reasoning in the previous section, we can begin the iterations with the origin function $\Phi_1(x) \equiv 0$. From (50) and for the numeric values previously assumed for ratios $EJ/\bar{\mu}$ and $\dot{p}/\bar{p}$ we find that $a_1 = 0$ and so

$$b_1 = \gamma\bar{Q} = 0{\cdot}02\bar{Q} \tag{54}$$

which corresponds to the elastic solution, as may be observed by placing (54) in (53) and deducing for $n = 1$ the expression:

$$\dot{Q}_2(x) = \tfrac{3}{8}\dot{p}lx - \frac{\dot{p}x^2}{2} = \dot{Q}^f(x) \tag{55}$$

Substituting (55) in (50) and proceeding with the iteration, we obtain the following values for b_n:

$$\begin{aligned} b_2 &= 0{\cdot}0196469\bar{Q} \\ b_3 &= 0{\cdot}0196591\bar{Q} \\ b_4 &= 0{\cdot}0196587\bar{Q} \end{aligned} \tag{56}$$

whose convergence on the numeric value (45) previously evaluated by the analogic method may be regarded as sufficiently rapid.

6. EXTENSIONS

When we come to consider incremental constitutive laws of the associated type with work-softening, we must always make *inter alia* one or more assumptions that limit or condition the work-hardening coefficient $\mu(x)$. This is because the assumption regarding the sign of the coefficient ($\mu(x) < 0$) alone is not sufficient to ensure uniqueness of solution of the incremental problem, as is quickly evident in particular cases or, more generally, from the discussion of equation (16).

To be able to extend the chain of reasoning so far expounded to the case of work-softening we must therefore make an additional assumption. For this purpose we may require that $\mu(x)$ should comply with the inequality:

$$\int_{V_p} \mu(x) \,.\, \dot{\lambda}^2(x)\,\mathrm{d}V - \iint_{V_p} Z(x, x') \,.\, \dot{\lambda}(x) \,.\, \dot{\lambda}(x')\,\mathrm{d}V\,\mathrm{d}V' > 0 \tag{57}$$

for any $\dot{\lambda}(x) \not\equiv 0$, a condition that in [4] was found to be sufficient both for uniqueness of the solution and for the general stability of the system.

Represent the condition of plasticity in the space of components $q_i(x)$ with regular surface $\Psi(q_i) = 0$ and define function $g[\dot{\lambda}(x')]$ in the following way:

$$\text{for } \Psi[q_{i0}(x')] \geq 0 \begin{cases} \text{if } \dot{\lambda}(x') \geq 0 & g[\dot{\lambda}(x')] = \dot{\lambda}(x') + h(x') \quad (58) \\ \text{if } \dot{\lambda}(x') < 0 & g[\dot{\lambda}(x')] = 0 \quad (59) \end{cases}$$

$$\text{for } \Psi[q_{i0}(x')] < 0 \begin{cases} \text{if } \dot{\lambda}(x') < 0 \quad \text{or } \dot{\lambda}(x') > 0 & g[\dot{\lambda}(x')] = 0 \quad (60) \\ \text{and } \Psi[q_{i0}(x') + \dot{q}_i(x')] < 0 & \\ & \\ \text{if } \dot{\lambda}(x') \geq 0 & g[\dot{\lambda}(x')] = \dot{\lambda}(x') + \bar{f}[q_{i0}] + h(x') \quad (61) \\ \text{and } \Psi[q_{i0}(x') + \dot{q}_i(x')] > 0 & \end{cases}$$

where $\bar{f}[q_{io}]$ is known like $\bar{f}[Q_{io}]$ in (11) (Fig. 2). Now as $\mu(x) < 0$, equation (16) is transformed into the analogous expression

$$\lambda(x) - \int_V \frac{Z(x, x')}{\mu(x)} \cdot g[\lambda(x')] \, dV' = 0 \tag{62}$$

for which the previous reasoning regarding the applicability and convergence of the iterative method (25') still hold good. Further, because of the additional hypothesis (57), the sufficient conditions for convergence of the simple iteration method, are now satisfied.

Indeed, given $\dot{u}(x) = \dot{\lambda}(x)\sqrt{[\mu(x)]}$, (57) is transformed in to the condition

$$\int_{V_p} \dot{u}^2(x) \, dV - \iint_{V_p} \frac{Z(x, x')}{\sqrt{[\mu(x)]} \cdot \sqrt{[\mu(x')]}} \dot{u}(x) \cdot \dot{u}(x') \, dV \, dV' > 0 \tag{63}$$

hence, on the condition for function $\dot{u}(x)$:

$$\int_{V_p} \dot{u}^2(x) \, dV = 1 \tag{64}$$

and for known extremum properties of the eigenvalues, we obtain (Γ_1 denoting the eigenvalue of least modulus):

$$\frac{1}{|\Gamma_1|} = \max \left| \iint_{V_p} \frac{Z(x, x')}{\sqrt{[\mu(x)]} \cdot \sqrt{[\mu(x')]}} \dot{u}(x) \cdot \dot{u}(x') \, dV \, dV' \right| < 1 \tag{65}$$

and hence

$$|\Gamma_i| > 1 \tag{22}$$

coincident with the sufficient condition (22) for convergence of the simple iteration method.

Lastly, regarding the possibility of a more general extension including the behavior of perfectly elastic–plastic materials ($\mu(x) = 0$), we would point out that a singularity arises in (16) that gives rise to a number of difficulties. There are some useful pointers in the theory of singular integral equations [14] to which the reader is referred for a fuller study of the subject.

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Абстракт—Преобразывается классический подход постепенно нарастающей задачи упруго-пластических конструкций, с элементарно упрочняющими составными элементами, выраженный вф орме интегрального уравнения, в какое либо новое, свободное от условий, для неизвестной постепенно нарастающей пластической деформации.

Работа использует некоторые итерационные методы теории интегральных уравнений и обсуждает условия одномерной сходимости последовательных функций, которые приближают к постепенно нарастающему распределению напряжений.

В заключение, показано, что рассматриваемые интегральные методы можно использовать для случая конструкций с смагчающимися составными элементами. Условие, которое обыно сопуствуея исследованию этих систем, совместное с условием сходимости одного из обсуждаемых итерационных методов.